Optimization of Openings in Plates under Plane Stress

Surendra K. Dhir*
David W. Taylor Naval Ship Research and Development Center, Bethesda, Maryland

An analytical/numerical procedure for optimizing certain geometrical and material aspects in the design of openings for both isotropic and anisotropic plate structures under in-plane loading is described. The procedure is based on first developing symbolic boundary-stress expressions as functions of opening geometry, plate material, amount of reinforcement, and specified loading. These stress expressions are then used to construct a meaningful objective function. Minimization of this function with respect to the opening geometry, the plate material constants, and the amount of reinforcement gives the desired optimum values. Four typical examples are included to demonstrate the procedure: 1) optimization of the shape of a square-like opening in a large isotropic plate, 2) optimization of the amount of reinforcement for a circular opening in a large isotropic plate, 3) optimization of the amount of reinforcement for a square-like opening in a large isotropic plate, and 4) optimization of the material constants for a circular opening in a large orthotropic plate. Results for these problems with several types of boundary loads are included.

Introduction

ONE of the most important problems in the design of plate members of ship and aircraft structures is to minimize the stress concentrations due to the presence of holes and other structural discontinuities. The literature contains a great deal of information on this problem; however, most of this information is concerned with direct determination of stresses for given geometry and load conditions. The use of reinforcements at boundaries of these holes is the most common technique for reducing the stress concentrations. The amount of reinforcement, in most cases, is determined by somewhat arbitrary and overly conservative decisions regarding the percentage reduction it produces in the boundary-stress maximums. Thus, the entire opening design process, i.e., from determining the shape of the opening to computing the amount of its reinforcement, seems to lack a rational basis or criterion for helping the structural designer. Some recent investigations[1-9] have attempted to optimize the shape of the unreinforced holes and/or discontinuities both experimentally and analytically. Experimental results of Durelli and Rajaiah[1] were obtained by step-by-step machining of hole boundaries in a photoelastic model until the tensile and compressive boundary stresses were approximately constant. This technique, therefore, calls for a series of experiments for each unique situation. The harmonic hole concept of Bjorkman and Richards[2,3] appears to have limited application as it results in somewhat impractical hole shapes. Schnack's procedure,[4] which in a way is the numerical equivalent of Durelli and Rajaiah's experimental approach, appears to be presently limited to unreinforced holes. Neuber's work[5,6] is, of course, well-known and forms the basis for many other investigations for direct stress analysis of boundaries involving notches and holes. A few other researchers[7-9] also have studied similar problems related to optimization. Some investigators[4-6] have concluded that uniform tangential stress at the notch boundary, in general, would lead to the smallest stress concentrations. This appears to be the case in the results described in this paper.

The present paper is a step toward a practical design procedure in that the optimization technique developed permits the designer to address the determination of the geometry of a given type of hole (e.g., square-like), the amount of boundary reinforcement required, and finally the material constants in the case of anisotropic base plates.

The procedure is based on first developing symbolic boundary stress expressions as functions of opening geometry, plate material, amount of reinforcement, and specified loading. These stress expressions are then used to construct a meaningful objective function by using one of the several available failure theories. Minimization of this objective function with respect to the opening geometry, the plate material constants, and the amount of reinforcement gives the desired optimum values for these design parameters.

The several mathematical formulations of this problem and their solutions contained enormously long and complex algebraic expressions. Manipulations of these expressions and their numerical solutions were considerably simplified by the use of MACSYMA, a symbolic manipulation language developed at MIT[10] and in regular use at DTNSRDC.

Optimization Procedure

In his paper, Schnack[4] bases his finite element procedure on the hypothesis of a constant tangential stress distribution for obtaining minimum notch stress. If one dispenses with the limitations on the geometry of the variational domain, trivial solutions could result in elimination of the notch. Imposition of certain geometrical restrictions would help circumvent this problem and would also provide for the designer a specification of the general overall opening shape (e.g., square-like) determined from other practical considerations. In view of these restrictions, the original hypothesis of a constant tangential stress should be modified to require "most uniform tangential stress" along the opening boundary.

The use of the strain energy density distribution V_0 around the opening boundary was found to be a relatively convenient and effective way of optimizing the geometry of the opening and the amount of reinforcement required, as well as the material constants if the plate is anisotropic. The quantity V_0 is integrated around the opening boundary to obtain an integral $I = \oint V_0 d\beta$ (where β is the curvilinear coordinate around the hole boundary). This integral, which, in general, represents the strain energy in a thin region of variable thickness around the opening, is the desired objective function. For the general problem, I would be a function of the material constants E_1, E_2, ν_1, and G of the anisotropic plate; of the coefficients m_n of the conformal mapping function;

Presented as Paper 82-0717 at the AIAA/ASME/ASCE/AHS 23rd Structures, Structural Dynamics and Materials Conference, New Orleans, La., May 10-12, 1982; submitted May 12, 1982; revision received Nov. 22, 1982.

*Supervisory Physicist.

and of the cross-sectional area A of the boundary reinforcement. Numerical optimization of I, a multivariate function, with some linear constraints appears to be a possible approach. However, as a first step, practical results were obtained by separately analyzing the general problem from three rather natural vantage points, i.e., geometry, amount of reinforcement, and material constants. Since, in all three cases, V_0 consists of sums of quadratic forms of stresses, this optimization procedure is based on minimization of the L^2 norm of stresses, a special case of the so-called L^p norms in a continuous function space. Minimization of other norms such as the L^1 norm (related to the absolute values of stresses) and the L^∞ norm (related to the maximum boundary stress values) was found to be unsuitable for the present formulation. A more detailed discussion is given in Ref. 11.

In contrast to minimization of I, minimization of other quantities such as the actual boundary strain energy V in a thin region of uniform thickness appeared to lead to attenuation of the peaks in the distribution of $V_0(\mathrm{d}s/\mathrm{d}\beta)$ (where $\mathrm{d}s$ is the arc length of a boundary element) rather than that of V_0. This, however, should be expected because the boundary-strain energy V is given by

$$\oint V_0 \mathrm{d}s \quad \text{or} \quad \int_0^{2\pi} V_0 \frac{\mathrm{d}s}{\mathrm{d}\beta} \mathrm{d}\beta$$

and in general, minimization of an integral of the type

$$\int_0^{2\pi} Q \mathrm{d}\beta$$

is likely to yield an attenuated Q distribution as a function of β.

Examples

A. Unreinforced Openings

A conformal mapping function represented by

$$z(\zeta) = \zeta + \frac{m_1}{\zeta} + \frac{m_3}{\zeta^3} + \frac{m_5}{\zeta^5} + \frac{m_7}{\zeta^7} \tag{1}$$

can map a unit circle in the ζ plane ($\zeta = e^{\alpha + i\beta}$) onto a hole with at least two axes of symmetry in the z plane ($z = x + iy$). This mapping function is chosen only to discuss the more practical opening shapes, otherwise more general functions may also be used. With standard procedures for solving boundary-value problems of plane elasticity, it can be shown that, for a large isotropic plate containing an opening described by Eq. (1) and subject to a biaxial plane stress field at infinity, the tangential stress σ_β at the opening boundary is given by[11]

$$\sigma_\beta = 4Re\left\{S + \sum_{n=1}^{4} \frac{P_n}{\sigma^2[1-(Q_n/\sigma^2)]}\right\} \tag{2}$$

where the P_n are defined as follows:

$$P_1 = -\frac{Q_1^4 S - Q_1^3 a_1 - 3Q_1^2 a_3 - 5Q_1 a_5 - 7a_7}{(Q_2 - Q_1)(Q_3 - Q_1)(Q_4 - Q_1)}$$

P_2, P_3, and P_4 can be obtained by cyclic permutation of subscripts of Q_n in the order 1,2,3,4 in the foregoing expression. a_1, a_3, a_5, and a_7 are defined as constants in the two holomorphic functions used to describe the plane stress field, and are included in Ref. 11. p and q represent the uniform stresses at infinity at an angle θ to the x and y axes, see Fig. 1. Furthermore, $S = (p+q)/4$ and $D = -(p-q)e^{-2i\theta}/2$. σ is the value of ζ at the hole boundary and Q_n are related to m_n by

$$\begin{aligned} m_1 &= \sum_{i=1}^{4} Q_i \\ m_3 &= -\frac{1}{3}\sum_{i=1}^{3}\sum_{j=i+1}^{4} Q_i Q_j \\ m_5 &= \frac{1}{5}\sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4} Q_i Q_j Q_k \\ m_7 &= -\frac{Q_1 Q_2 Q_3 Q_4}{7} \end{aligned} \tag{3}$$

Now, Eq. (2) can be used to show that

$$I_0 = \frac{1}{16\pi}\int_0^{2\pi} \sigma_\beta^2 \mathrm{d}\beta = 2S^2 + \sum_1^4 \sum_1^4 \frac{P_m P_n}{1 - Q_m Q_n} \tag{4}$$

where I_0 is $I/(16\pi)$. Equation (4) is general enough to consider geometrical optimization of square and rectangular openings. For example, in the case of a square-like opening ($m_1 = m_5 = m_7 = 0$ and $m_3 \le 1/6$)

$$I_0 = -2S^2 + \frac{4S^2}{1-9m_3^2} + \frac{D^2}{(1-m_3)^2(1-9m_3^2)} \tag{5}$$

The first derivative of this integral with respect to m_3 yields the following condition for extremum values of I_0.

$$\begin{aligned} &36S^2 m_3^4 - 108S^2 m_3^3 + 18(D^2 + 6S^2)m_3^2 \\ &-9(D^2 + 4S^2)m_3 - D^2 = 0 \end{aligned} \tag{6}$$

The symbolic solution of Eq. (6) was obtained as a function of S and D, but it is not included here. For specific cases of interest, with a linear constraint $|m_3| \le 1/6$, the values of m_3 which optimize the geometry of a square-like hole are given in Table 1. A detailed discussion of these results can be found in Ref. 11, and the optimized shape at $m_3 = -0.05$ including two other shapes for comparison are shown in Fig. 1.

The case discussed so far dealt with the mapping function containing only one term. Inclusion of additional terms led to unwieldy algebraic expressions; therefore, a direct numerical minimization approach was followed to determine the m_n. For example, it was found that in case 2 of Table 1, $m_3 = -0.05$ and $m_7 = 0.0035$ produced an $I = 0.3623$. This meant an improvement over case 2, even though the actual difference is small. The optimization procedure described is general enough to include any number of m_n.

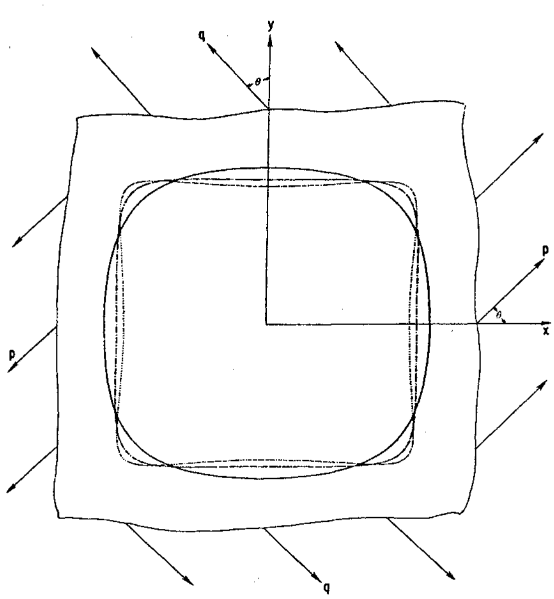

Fig. 1 Optimized square-like opening. --- $m_3 = -1/6$; -·-·- $m_3 = -1/8$; —— $m_3 = -0.05$.

B. Reinforced Openings

An opening of general shape in a large elastic plate of isotropic material of unit thickness is reinforced by a thin member of cross-sectional area A capable of withstanding axial forces only. Then, if the opening is mapped onto a unit circle by a function $z(\zeta)$, the equivalence between complex forces in the reinforcement and those in the plate at the opening boundary ($\zeta=\sigma$) is given by[12]

$$\phi(\sigma)+\frac{z(\sigma)}{\overline{z'(\sigma)}}\overline{\phi'(\sigma)}+\overline{\psi(\sigma)}=\sigma P\sqrt{\frac{z'(\sigma)}{\overline{z'(\sigma)}}}+C \tag{7}$$

where $\phi(\sigma)$ and $\psi(\sigma)$ are the values of the functions $\phi(\zeta)$ and $\psi(\zeta)$ at the opening boundary, P is the axial force in the reinforcement, σ equals $e^{i\beta}$, and C is an arbitrary constant. Details of the solution of Eq. (7) and the associated displacement continuity conditions are given in Ref. 12. This solution makes possible symbolic determination of the stresses σ_α, σ_β, and $\tau_{\alpha\beta}$ at the opening boundary. In general, these stresses would be functions of S, D, and A, and of the coefficients of the mapping function. The strain energy density V_0 is now given by

$$V_0=\frac{1}{2E}\left[\sigma_\alpha^2+\sigma_\beta^2-2\nu\sigma_\alpha\sigma_\beta+2(1+\nu)\tau_{\alpha\beta}^2\right] \tag{8}$$

In the case of a circular opening, since all the coefficients m_n are zero, I can be computed as

$$I=V=\frac{8\pi}{E}\left[\frac{2(1+0.91A^2)S^2}{(1+1.3A)^2}+\frac{(1+11.31A^2)D^2}{(1+3.3A)^2}\right] \tag{9}$$

where ν has been assumed to be 0.3. Note that in this case, the integral I is the same as the strain energy around the opening boundary. Table 2 gives the values of A which minimize the integral I for several load cases of interest. Also included for comparison are the values of A based on the minimization of other relevant integrals, see Ref. 12.

Table 1 Optimum values of m_3 for a square-like opening

Number	Loading	m_3	I_0	σ_{max}
1	$p=1$, $q=1$	0	0.5	2.00
2	$p=1$, $q=0$	−0.05	0.3627	2.47
3	$p=2$, $q=1$	−0.01	1.3723	4.75
4	$p=1$, $q=1$	−0.09	0.9077	3.07

The values of A refer to the ratio of the area replaced as the reinforcement to the area removed for the opening. Thus in the uniaxial case, if approximately 44.3% of the area removed for the opening is replaced as reinforcement, the resulting stress distribution would be very desirable, i.e., one that would result in a relatively smooth boundary-strain energy density distribution. Various stress expressions as a function of A, the area of reinforcement, can be found for a circular opening in Ref. 13.

In the case of a square opening, the coefficient m_3 of the mapping function, Eq. (1), was assigned a nonzero value of -0.05. This value corresponds to the optimized square shape determined earlier. The algebraic manipulations required to solve Eq. (7) in this case were considerably more involved than the circular opening because the right side of Eq. (7) is irrational for noncircular openings. Thus, long Taylor series expansions had to be used to arrive at the solution. These series expansions maintained an accuracy of 1 in 10^6 at the opening boundary. Table 3 includes the minimized values of I and the corresponding values of A for the uniaxial case, i.e., $p=1$, and $q=0$. Figure 2 shows the stresses σ_α, σ_β, and $\tau_{\alpha\beta}$ for a reinforced square-like opening with $A=0.412$ and also, for comparison, the equivalent stress σ_E based on the strain energy theory of failure and the value of σ_β for the unreinforced opening.

C. Anisotropic Plates

The expressions for tangential stress at the boundary of a circular opening in a large orthotropic plate, subjected to the following two loading conditions, are taken from Lekhnitski.[14]

1) Normal pressure p distributed uniformly along the opening edge

$$\sigma_\theta=p\frac{E_\theta}{E_1}\left[-k+n(\sin^2\theta+k\cos^2\theta)+(1+\mu_1)(1+\mu_2)\sin^2\theta\cos^2\theta\right] \tag{10}$$

2) Tension p at infinity at an angle ϕ to a principal direction of elasticity

$$\sigma_\theta=p\frac{E_\theta}{E_1}\{[-\cos^2\phi+(k+n)\sin^2\phi]k\cos^2\theta+[(1+n)\cos^2\phi-k\sin^2\phi]\sin^2\theta-n(1+k+n)\sin\phi\cos\phi\sin\theta\cos\theta\} \tag{11}$$

In these equations μ_1 and μ_2 are complex parameters such that the following relations hold.[14]

$$k=-\mu_1\mu_2=\sqrt{(E_1/E_2)}$$
$$m=-\mu_1^2-\mu_2^2=(E_1/G)-2\nu_1$$
$$n=-i(\mu_1+\mu_2)=\sqrt{2k+m} \tag{12}$$

Table 2 Optimum values of reinforcement for a circular hole subject to various types of loads

Load cases \ Attenuated quantity	Strain energy density V_0	Strain energy density A	Distortion energy density V_D	Distortion energy density A	Volume energy density V_v	Volume energy density A	Max shear stress squared T	Max shear stress squared A
Isotropic $S=0.5$; $D=0$	$\frac{2.8\pi}{2E}$	1.4286	$\frac{1.733\pi}{2E}$	1.4286	$\frac{1.067\pi}{2E}$	—[a]	0	1.4286
Uniaxial $S=0.25$; $D=-0.5$	$\frac{3.074\pi}{2E}$	0.4429	$\frac{2.584\pi}{2E}$	0.4033	$\frac{0.349\pi}{2E}$	∞[b]	2.717π	0.3422
Pure shear $S=0$; $D=-1$	$\frac{8.15\pi}{2E}$	0.2918	$\frac{7.072\pi}{2E}$	0.2602	$\frac{0.331\pi}{2E}$	∞[b]	8π	0.2127

[a] Indicates that V_v is independent of A. [b] Infinity refers to a rigid reinforcement.

where E_I and E_2 are Young's moduli, ν_I one of the Poisson ratios, and G the shear modulus (for principal directions). Equations (12) make it possible to express E_I/E_θ as

$$E_I/E_\theta = \sin^4\theta + m\sin^2\theta\cos^2\theta + k^2\cos^4\theta \qquad (13)$$

Equations (10) and (11) can also be transformed so that only two material parameters, k and m, occur in the expressions for σ_θ.

For Eq. (10), i.e., for a uniformly loaded opening, it can be shown that, if $m = 1 + k^2$, the stress σ_θ becomes uniformly unity around the opening boundary. As expected, a numerical computation for minimizing the value of the integral I led to the same conclusion. Obviously, an infinite number of combinations of the material constants E_I, E_2, ν_I, and G is possible which fulfill the condition $m = 1 + k^2$ for obtaining the optimum stress field.

Linear combinations of Eq. (11) with different values for ϕ can be used to represent several loading cases. For example, a plate compressed by a unit pressure in a principal direction, which cannot expand in the transverse direction, can be represented by superimposing $p = -1$, $\phi = 0$, and $p = -\nu_2$, $\phi = \pi/2$, where ν_2 $(= \nu_I/k^2)$ is the Poisson's ratio in the other principal direction. Integral I was numerically computed for a number of values of k and n to determine the minimum points for several load cases. Some of these minimum points representing optimum stress fields are included in Table 4.

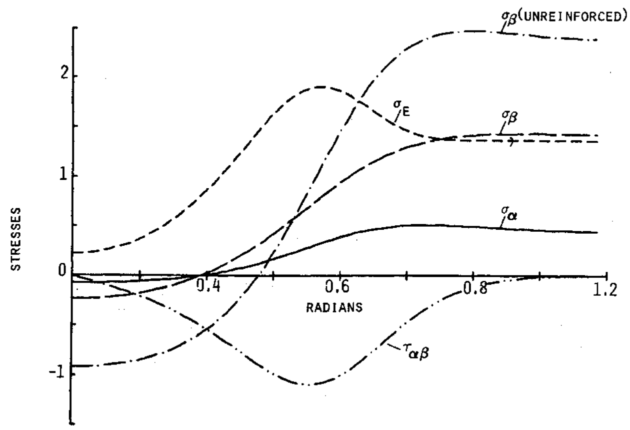

Fig. 2 Various boundary stresses around a square-like opening with optimum reinforcement.

Table 3 Minimized values of I at optimum values of A for a square opening

A	$\frac{2E}{16\pi} \times I$	A	$\frac{2E}{16\pi} \times I$
0.000001	0.36268	0.5	0.19324
0.1	0.24854	0.6	0.19671
0.2	0.20936	0.7	0.20120
0.3	0.19546	0.8	0.20610
0.4	0.19199	0.9	0.21109
0.412	0.19195	1.0	0.21600

Table 4 Optimum values of E_I/E_2 for an orthotropic plate with a circular hole and $m = 11.6$

Load cases	Minimized integral, I	k^2 (E_I/E_2)	σ_{max}	Comments
Hydrostatic $p = q = 1$	25.132	10.6	2.00	Uniform
Uniaxial $p = 1$, $\phi = 0$	21.196	0.6	4.63	σ_{max} at $\theta = \pi/2$
Uniaxial $p = 1$, $\phi = \pi/2$	19.348	2.8	3.35	σ_{max} at $\theta = 0$
Pure shear $p = -q = 1$ $\phi = \pi/4$	54.304	1.6	−3.64	σ_{max} at $\theta = 70$ deg
Biaxial $p = -1$, $q = -\nu_2$	19.424	0.7	−4.40	σ_{max} at $\theta = \pi/2$

Conclusions

A procedure based on the minimization of a certain stress integral has been developed which can effectively be used to 1) optimize the geometry of a class of hole shapes in structures consisting of large plates, 2) show that for openings in large plates an optimum amount of reinforcement actually exists which corresponds to a minimum value of a strain energy related integral and to determine the amount of this optimum reinforcement, and 3) determine the material constants such that the stress field around a circular opening in an orthotropic plate is an optimum.

References

[1] Durelli, A. J. and Rajaiah, K., "Quasi-Square Hole with Optimum Shape in an Infinite Plate Subjected to In-Plane Loading," Oakland University ONR Rept. 49, Jan. 1979.

[2] Bjorkman, G. S. and Richards, R., "Harmonic Holes–An Inverse Problem in Elasticity," *ASME Journal of Applied Mechanics*, Vol. 43, Sept. 1976, pp. 414-418.

[3] Bjorkman, G. S. and Richards, R., "Harmonic Holes for Nonconstant Field," *ASME Journal of Applied Mechanics*, Vol. 46, Sept. 1979, pp. 573-576.

[4] Schnack, E., "An Optimization Procedure for Stress Concentration by the Finite-Element Technique," *International Journal for Numerical Methods in Engineering*, Vol. 14, Jan. 1979, pp. 115-124.

[5] Neuber, H., *Kerbspannungslehre Grundlagen fur genaue Festigkeitsberechnung*, 2nd ed., Springer-Verlag, Berlin, 1958.

[6] Neuber, H., "Zur Optimierung der Spannungskonzentration," *Continuum Mechanics and Related Problems of Analysis*, Moskau, Nauka, 1972, pp. 375-380.

[7] Cherepanov, G. P., "Inverse Problems of the Plane Theory of Elasticity," *Journal of Applied Mathematics and Mechanics* (translation of *Soviet Journal Prikladnaya Matematika i Mechanika*), Vol. 38, 1974, pp. 963-979.

[8] Banichuk, N. V., "On a Variational Problem with Unknown Boundaries and the Determination of Optimal Shapes of Elastic Bodies," *Journal of Applied Mathematics and Mechanics* (translation of *Soviet Journal Prikladnaya Matematika i Mechanika*), Vol. 39, June 1975, pp. 1082-1092.

[9] Wheeler, L., "On the Role of Constant-Stress Surfaces in the Problem of Minimizing Elastic Stress Concentration," *International Journal of Solids and Structures*, Vol. 12, Nov. 1976, pp. 779-789.

[10] Mathlab Group, "MACSYMA Reference Manual," Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, Mass., Version 9, Dec. 1977.

[11] Dhir, S. K., "Optimization in a Class of Hole Shapes in Plate Structures," *ASME Journal of Applied Mechanics*, Vol. 48, Dec. 1981, pp. 905-908.

[12] Dhir, S. K., "Optimization of Reinforcement for a Class of Openings in Plate Structures," DTNSRDC/CMLD Rept. 81-25, Sept. 1981; also, *ASME Journal of Applied Mechanics*, Vol. 49 Dec. 1982, pp. 416-420.

[13] Radok, J. R. M., "Problems of Plane Elasticity for Reinforced Boundaries," *ASME Journal of Applied Mechanics*, Vol. 77, June 1955, pp. 249-254.

[14] Lekhnitskii, S. G., *Anisotropic Plates*, translated by S. W. Tsai and T. Cheron, Gordon and Breach Science Publishers, New York, 1968.